

# Club guessing and the universal models

Mirna Džamonja  
 School of Mathematics  
 University of East Anglia  
 Norwich, NR4 7TJ, UK

M.Džamonja@uea.ac.uk

<http://www.mth.uea.ac.uk/people/md.html>

February 2, 2008

## Abstract

We survey the use of club guessing and other pcf constructs in the context of showing that a given partially ordered class of objects does not have a largest, or a universal element.<sup>1</sup>

## 0 Introduction

A natural problem in mathematics is the following: given some partially ordered or a quasi-ordered set or a class, is there the largest element in it. An aspect of this question appears in the theory of order where one concentrates on the properties of the set and the partial order, ignoring the properties of the individual elements of the set. A very different view of this question is obtained when one takes the point that it is the structure of individual elements that is of interest. An instance of this is the question of universality. Here we are given a class or a set of objects and a notion of embedding between them, and we ask if there is an object in the class that

---

<sup>1</sup>The author thanks EPSRC for their support on as Advanced Fellowship. AMS 2000 Classification: 03C55, 03E04, 03C45.

Keywords: universal models, club guessing.

embeds all the others. To simplify our exposition here we shall always assume that we are working with a set of objects and we shall discuss the smallest cardinality of the subset of that set that has the property of embedding all the other objects in the set. We shall refer to this question as *the universality problem*, the class and the embedding to which the problem refers will always be clear from the context. The number mentioned above will be then referred to as *the universality number*.

Instances of the universality problem have been of a continuous interest to mathematicians, especially those studying the mathematics of the infinite—even Cantor’s work on the uniqueness of the rational numbers as the countable dense linear order with no endpoints is a result of this type. For some more recent examples see [1], [12] and [45]. Apart from its intrinsic interest the universality problem has an application in model theory, more specifically in classification theory. There it is used to distinguish between various kinds of unstable theories. For more on this programme the reader may consult the introduction to [9] and some of the main results will be mentioned in §3. A good source about the classical results about universality is the introduction to [15]. The study of the universality problem can naturally be divided into ‘the positive’ and ‘the negative’ part. On the positive side one tries to show that the universality number has at most the given value, for example 1. Proofs here are often explicit constructions of universal objects, or forcing constructions, see [4], [25] or [8]. On the negative side one does the opposite, showing that the universality number is at least a given value. In this paper we shall concentrate on the negative side of the universality problem, in particular on the instances of it that are obtained using a specific method that has appeared as a consequence of the discovery of the pcf theory, namely the club guessing method. There are several other methods that appear in the study of the negative side of the universality problem, notably the  $\sigma$ -functor of Stevo Todorćević [45], but here we shall only concentrate on the club guessing method. This method is due to Menachem Kojman and Saharon Shelah [15]. We shall start by recalling the basic principles behind it and then give some applications including the original one from [15] to linear orders. Let us also note that in this subject it is often not difficult to construct a specific universe of set theory in which the desired negative

universality result holds. For example, one can find in [15] a proof of the fact that when one forces over a model of  $GCH$  to add  $\lambda^{++}$  Cohen subsets to a regular cardinal  $\lambda$ , then in the resulting universe there is no universal graph of size  $\lambda$ , no universal linear order of size  $\lambda$ , or a universal model of any first order countable theory unstable in  $\lambda$ . The point of the negative universality results obtained by the club guessing method is that they are implications between a certain pcf statement and the desired negative universality result, so they hold in more than just one specifically constructed universe.

This paper is a survey of some of the existing techniques and results in this subject. Due to the extensive literature it has not been possible to mention all relevant results, so our apologies go to the authors of the many deserving papers which we failed to mention.

## 1 Invariants and linear orders

Let us start with an easy example of a negative universality result: we remind the reader of why it is that for an infinite cardinal  $\lambda$  there is no well ordering of size  $\lambda$  to which there is an order-preserving embedding from any well order of size  $\lambda$ . The reason for this is that any well order of size  $\lambda$  is ordered in a type  $\zeta < \lambda^+$  and hence cannot embed ordinals larger than  $\zeta$ . This simple proof has three important elements: invariants, construction and preservation. Specifically, to each well order we have associated an *invariant*, namely its order type, then we observed that the invariant is *preserved*, in the sense that it can only increase under embedding, and finally we have *constructed* a family of well orders of size  $\lambda$  where many different values of the invariant are present (namely the ordinals in  $[\lambda, \lambda^+)$ ), so showing that no single well order of size  $\lambda$  can embed them all.

These same principles are present in many settings, for example [1], and in particular in the club guessing proofs. The matters of course tend to be more complex. We shall now show the original Kojman-Shelah example of the use of club guessing for a universality result about linear orders, cf. [15].

Let  $\lambda$  be a regular cardinal and let  $\mathcal{K}$  be the class of all linear orders whose size is  $\lambda$ . By identifying the elements of  $\mathcal{K}$  that are isomorphic to each other we obtain a set of size at most  $2^\lambda$ , which we shall call  $\mathcal{K}$  again. We

may without loss of generality assume that the universe of each element of  $\mathcal{K}$  is the  $\lambda$  itself. We are interested in the universality number of  $\mathcal{K}$ , where for our notion of the embedding we take an injective order preserving function. For future purposes let us also fix a ladder system  $\bar{C} = \langle C_\delta : \delta \in S \subseteq \lambda \rangle$ , such that each  $C_\delta$  is a club of the corresponding  $\delta$  and  $S$  is some stationary subset of the set of limit ordinals below  $\lambda$ . Let  $C_\delta = \langle \alpha^\delta : i < i^*(\delta) \rangle$  be the increasing enumeration.

Every member  $L$  of  $\mathcal{K}$  can be easily represented as a continuous increasing union  $L = \bigcup_{j < \lambda} L_j$  of linear orders of size  $< \lambda$ , and such a representation is of course not unique. Any such sequence  $\bar{L} = \langle L_j : j < \lambda \rangle$  is called a *filtration* of  $L$ . Next we shall define invariants for elements of  $L$ , but the definition of an invariant will depend both on the filtration  $\bar{L}$  and on the specified ladder system  $\bar{C}$ .

**Definition 1.1** Suppose that  $L, \bar{L}, \bar{C}$  are as above and  $\delta \in S$  is such that the universe of  $L_\delta$  is  $\delta$ . We define the *invariant*  $\text{inv}_{\bar{L}, \bar{C}}(\delta)$  as the set

$$\{i < i^*(\delta) : (\exists \beta \in L_{\alpha_{i+1}^\delta} \setminus L_{\alpha_i^\delta})[\{x \in L_{\alpha_i^\delta} : x <_L \beta\} = \{x \in L_{\alpha_i^\delta} : x <_L \delta\}]\}.$$

So with the notation of Definition 1.1, the invariant of  $\delta$  is a subset of  $i_\delta^*$  that codes the ‘reflections’ of  $\delta$  along the places in the filtration  $\bar{L}$  that are determined by  $\bar{C}$ . It is easy to check that the set of  $\delta$  for which the universe of  $L_\delta$  is  $\delta$  is a club of  $\lambda$ , so since  $S$  is stationary there is a club of  $\delta$  for which  $\text{inv}_{\bar{L}, \bar{C}}(\delta)$  is well defined. It is also easy to see that for any two filtrations  $\bar{L}$  and  $\bar{L}' = \langle L'_j : j < \lambda \rangle$  of  $L$  there is a club of  $\delta$  such that  $L_\delta = L'_\delta$ , hence the dependence of the invariant on the filtration is only up to a club. This is not the case with its dependence on the ladder system, and in fact only certain ladder systems are of interest to us:

**Definition 1.2** A ladder system  $\bar{C} = \langle C_\delta : \delta \in S \rangle$  is said to be a *club guessing sequence* iff for every club  $E$  of  $\lambda$  there is  $\delta \in S$  such that  $C_\delta \subseteq E$ .

Club guessing sequences were introduced by Shelah in [24] as a tool towards the development of pcf theory and have since found many applications in various contexts. There is also a number of variants of this concept, for example a number of interesting results about various kinds of club sequences

appears in Tetsuya Ishiu's thesis [13]. However, for the moment we shall concentrate on the simple club guessing mentioned above. An interesting question is when such sequences exist, and one of the most important theorems in this vein is the following:

**Theorem 1.3** [Shelah, [24]] Suppose that  $\kappa$  and  $\lambda$  are regular cardinals such that  $\kappa^+ < \lambda$ . Then there is a club guessing sequence of the form  $\bar{C} = \langle C_\delta : \delta \in S_\kappa^\lambda \rangle$ .

We have used the notation  $S_\kappa^\lambda$  to denote the set of  $\alpha < \lambda$  whose cofinality is  $\kappa$ . A club guessing sequence of the form appearing in Theorem 1.3 will be referred to as an  $S_\kappa^\lambda$ -club guessing sequence. Note that we may without loss of generality, by intersecting with a club of order type  $\kappa$  if necessary, assume that each  $C_\delta$  has order type  $\kappa$ . In fact, a stronger theorem than 1.3 was proved by Shelah, showing that in addition to this restriction on the order type the club guessing sequence above may be assumed to have a square-like property, in the terminology of [9] to be a *truly tight*  $(\kappa, \lambda)$  club guessing sequence:

**Theorem 1.4** [Shelah, [28]] Suppose that  $\kappa$  and  $\lambda$  are regular cardinals such that  $\kappa^+ < \lambda$ . Then there is a stationary set  $S \subseteq S_\kappa^\lambda$ , a club guessing sequence of the form  $\bar{C} = \langle C_\delta : \delta \in S \rangle$  and a sequence  $\bar{P} = \langle P_\alpha : \alpha < \lambda \rangle$  such that each  $P_\alpha \subseteq \mathcal{P}(\alpha)$  has size  $< \lambda$  and the sequences  $\bar{C}$  and  $\bar{P}$  interact in the sense that

$$\alpha \in C_\delta \setminus \sup(C_\delta \cap \alpha) \implies C_\delta \cap \alpha \in \bigcup_{\beta < \alpha} P_\beta.$$

A nicely written proof of Theorem 1.3 appears in [15]. Theorem 1.4 can be read off from the conjunction of the claims in §1 of [28], and a direct proof is given in [[37], 1.3.(a)]. Theorem 1.3 refers to cardinals  $\kappa, \lambda$  which have at least one successor gap,  $\kappa^+ < \lambda$ . It is natural to ask what happens at  $\lambda = \aleph_1$ . Clearly, in the presence of  $\clubsuit$  there is a  $S_{\aleph_0}^{\aleph_1}$ -guessing sequence. A theorem of Shelah (Theorem III 7.1 in [23]) shows that it is consistent to have  $2^{\aleph_0} = \aleph_2$  and that there is no  $S_{\aleph_0}^{\aleph_1}$ -club guessing sequence (in fact the theorem shows something stronger than what is being quoted here). This also follows from the conjunction of a result of Shelah in [25] which shows it consistent

to have the negation of  $CH$  and the existence of a universal linear order of size  $\aleph_1$ , and the Kojman-Shelah's theorem 1.5 which implies that in a model in which  $CH$  fails and there is a  $S_{\aleph_0}^{\aleph_1}$ -guessing sequence there cannot be a universal linear order of size  $\aleph_1$ . In the other direction, a consistency result showing how to add club guessing sequences with strong guessing properties was introduced by Péter Komjáth and Matthew Foreman in [11], who give a cardinal-preserving forcing which for any regular  $\lambda$  and a given stationary set  $S$  in  $\lambda$ , keeps  $S$  stationary and adds a sequence  $\langle C_\delta : \delta \in S \rangle$  such that for every club  $E$  of  $\lambda$  there is a club  $C$  such that for all  $\alpha \in C \cap S$  a non-empty final segment of  $C_\delta$  is included in  $E$ .

We shall now state and give a sketch of the proof of the Kojman-Shelah theorem on linear orders, using the notions of an invariant and a club guessing sequence introduced above. For the case of  $\kappa = \aleph_0$  we only need to use a  $S_\kappa^\lambda$  club guessing sequence, while the case  $\kappa > \aleph_0$  is handled using Theorem 1.4.

**Theorem 1.5** [Kojman-Shelah [15]] Suppose that  $\kappa$  and  $\lambda$  are regular such that  $\lambda < 2^\kappa$ . Further suppose that either

- (a)  $\kappa = \aleph_0, \lambda = \aleph_1$  and there is an  $S_{\aleph_0}^{\aleph_1}$ -club guessing sequence; or
- (b)  $\kappa^+ < \lambda$ .

Then there is no universal linear order of size  $\lambda$ , moreover the universality number of the class of linear orders of size  $\lambda$  is at least  $2^\kappa$ .

**Proof.** (sketch) The proof uses the method of Construction and Preservation. First let us fix a club guessing sequence  $\bar{C} = \langle C_\delta : \delta \in S \subseteq S_\kappa^\lambda \rangle$  and if  $\kappa > \aleph_0$  assume also that this sequence is chosen in conjunction with a sequence  $\bar{P}$  to form a truly tight  $(\kappa, \lambda)$ -guessing sequence. In particular we assume that the order type  $i^*(\delta)$  of  $C_\delta$  is always  $\kappa$ . Recall the notation  $\langle \alpha_i^\delta : i < \kappa \rangle$  for the increasing enumeration of  $C_\delta$ .

**Lemma 1.6** [Construction Lemma] For every  $A \subseteq \kappa$ , there is a linear order  $L_A$  and its filtration  $\bar{L}_A$  such that for a club  $C$  of  $\lambda$ , we have

$$(\delta \in C \ \& \ C_\delta \subseteq \delta) \implies \text{inv}_{\bar{L}_A, \bar{C}}(\delta) = A.$$

**Proof.** (sketch) The Construction Lemma has a simpler proof in the case (a) of the theorem, which is the case we shall prove. Then we shall comment on the changes needed to cover case (b). So assume that  $\kappa = \aleph_0$ ,  $\lambda = \aleph_1$  and  $\bar{C}$  is an  $S_{\aleph_0}^{\aleph_1}$ -club guessing sequence. Let  $A \subseteq \omega$  be given.

Recall that by a *cut* of a linear order we mean an initial segment of the order, and we say that the cut is *realised* if it has the least upper bound. If  $L \subseteq L'$  are linear orders and  $D$  is a cut of  $L$ , then a cut  $D'$  of  $L'$  *extends*  $D$  if  $D' \cap L = D$ . When speaking of a linear order  $(L, <_L)$  we may refer to cuts of  $L$  or cuts of  $<_L$ , as is more convenient for the context. By  $\eta$  we denote the order type of the rationals.

We shall define the order  $<_{L_A}$  on  $\omega_1$  by inductively defining a strictly increasing sequence  $\langle \gamma_i : i < \omega_1 \rangle$  of countable ordinals and defining  $<_i \stackrel{\text{def}}{=} <_{L_A} \upharpoonright \gamma_i$  at the step  $i$  of the induction. The requirements of this induction will be:

- (i) for every  $i < j$  there is a cut of  $<_i$  realised in  $\gamma_{j+1}$  and not realised in  $\gamma_j$ ;
- (ii) if  $i < j < k$  and  $D$  is a cut of  $\gamma_i$  realised in  $\gamma_j$  but not in  $\gamma_i$ , then there is a cut of  $D$  of  $\gamma_k$  that extends  $D$  and that is realised in  $\gamma_{k+1}$  but not in  $\gamma_k$ ;
- (iii) if  $D$  is a cut of  $\gamma_i$  realised in  $\gamma_{i+1}$  and not realised in  $\gamma_i$ , then the  $<_{i+1}$ -order type of  $\{x \in \gamma_{i+1} : x \text{ realises } D\}$  is  $\eta$ .

The starting point of the induction is  $\gamma_0 = \omega$  where we let  $<_0$  be  $\omega$  ordered in the order type of the rationals. At limit  $i$  we define  $\gamma_i$  to be the  $\sup_{j < i} \gamma_j$  and the order is defined as the union of the orders constructed so far.

At the stage  $\gamma_{i+1}$  we ask ourselves if  $i$  is a *good point* of the construction, which means that  $i$  is a limit ordinal and  $\gamma_i = i$ . If so we then ask if  $C_\delta$  consists of good points. If the answer to both of these questions is affirmative we proceed to define a sequence  $\langle D_n : n < \omega \rangle$  of cuts, such that each  $D_n$  is a cut of  $\alpha_n^i$  not realised in  $<_{\alpha_n^i}$  and  $D_n \subseteq D_{n+1}$ . In addition we require that  $D_n$  is realised in  $<_{\alpha_{n+1}^i}$  iff  $n \in A$ . If  $A = \emptyset$  we let  $D_0$  be a cut of  $<_0$  not realised in  $i$ , which exists as there are  $2^{\aleph_0}$  cuts of  $<_0$  and  $i$  is countable. We let  $D_n$  be the extension of  $D$  to  $\alpha_n^i$ , so  $D_n \stackrel{\text{def}}{=} \{x < \alpha_n^i : (\exists y \in D)(x <_{\alpha_n^i} y)\}$ . Otherwise, let  $n_0$  be the first element of  $A$  and let  $D_{n_0}$  be a cut of  $\alpha_{n_0}^i$  that is realised in  $\alpha_{n_0+1}^i$  and not in  $\alpha_{n_0}^i$ . For  $m < n_0$  let  $D_m = D_{n_0} \cap \alpha_m^i$ . If

$A = \{n_0\}$ , then since there is the order type  $\eta$  of elements of  $\alpha_{n_0+1}^i$  that realise  $D_{n_0}$  (by requirement (ii)), there is a cut  $D$  of  $i$  that extends  $D_{n_0}$  and that is not realised in  $i$ . For  $n > n_0$  we let  $D_n$  be  $D \cap \alpha_n^i$ . Otherwise, let  $n_1 = \min A \setminus (n_0 + 1)$  and let  $D_{n_1}$  be a cut of  $\alpha_{n_1}^i$  extending  $D_{n_0}$ , which is realised in  $\alpha_{n_1+1}^i$  and not in  $\alpha_{n_1}$ , which exists by (iii) above. Then we continue similarly to the previous case. In any case, we have constructed the increasing sequence of cuts as required, and letting  $D^*$  be their union we then let  $i$  realise  $D^*$  in  $<_L \upharpoonright (i + 1)$ . We extend the order by transitivity.

Now we still have to assure that the requirements of the induction are preserved, which can be done by amalgamating countably many ordinals to  $i + 1$  in the way requested by the requirements. The sup of all these ordinals is then defined to be  $\gamma_{i+1}$ . If  $i$  is not a good point we do not have to take special care of  $i$  but instead proceed just as in this paragraph.

At the end of the induction we let  $<_{L_A}$  be the union of the orders constructed,  $\bar{L}_A = \langle <_i : i < \omega_1 \rangle$  and  $E$  a club of good points of the construction. The construction was made so that at any  $\delta \in E$  the required invariant is guaranteed to be achieved only if  $C_\delta \subseteq E$ .

This finishes the proof of the Construction Lemma in the case  $\kappa = \aleph_0$ . For larger  $\kappa$  things become more complex, as one also has to handle the limit points of cofinality  $< \kappa$ . This is a difficulty familiar from classical constructions, such as that of a Suslin tree from a  $\diamond$ , where one in addition uses a  $\square$  sequence at cardinals larger than  $\aleph_1$ . In this case the construction can be carried through thanks to the square-like properties of a truly tight guessing sequence. See [15] for details. ★<sub>1.6</sub>

We also need the Preservation Lemma.

**Lemma 1.7** [Preservation Lemma] Suppose that  $L$  and  $L'$  are linear orders with universe  $\lambda$  and with filtrations  $\bar{L}$  and  $\bar{L}'$  respectively, while  $f : L \rightarrow L'$  is an order-preserving injection. Then there is a club  $E$  of  $\lambda$  such that for every  $\delta \in S_\kappa^\lambda$  satisfying  $C_\delta \subseteq E$ , we have

$$\text{inv}_{\bar{L}, \bar{C}}(\delta) = \text{inv}_{\bar{L}', \bar{C}}(f(\delta)).$$

**Proof.**(sketch) We start by defining a model  $M$  with universe  $\lambda$ , order relations  $<_L$ ,  $<_{L'}$  and  $<$  (the ordinary order on the ordinals) and the function  $f$ .



Let  $E$  be a club of  $\delta < \lambda$  such that  $\delta \in E$  implies that  $M \upharpoonright \delta \prec M$  and the universe of both  $L_\delta$  and  $L'_\delta$  is  $\delta$ . Suppose that  $\delta \in E$  is such that  $C_\delta \subseteq E$ , and we shall prove that  $\text{inv}_{\bar{L}, \bar{C}}(\delta) = \text{inv}_{\bar{L}', \bar{C}}(f(\delta))$ . The more difficult direction of the proof is the inclusion  $\subseteq$ . So suppose that  $i \in \text{inv}_{\bar{L}, \bar{C}}(\delta)$ , hence there is  $\beta \in L_{\alpha_{i+1}^\delta} \setminus L_{\alpha_i^\delta}$  satisfying that  $\{x <_L \beta : x \in L_{\alpha_i^\delta}\} = \{x <_L \delta : x \in L_{\alpha_i^\delta}\}$ . We would like to claim that  $f(\beta)$  witnesses that  $i \in \text{inv}_{\bar{L}', \bar{C}}(f(\delta))$ , and it does follow from the choice of  $E$  and the fact that  $C_\delta \subseteq E$  that

$$\{y <_{L'} f(\beta) : y \in f''(L_{\alpha_i^\delta})\} = \{y <_{L'} f(\delta) : y \in f''(L_{\alpha_i^\delta})\}.$$

However, the problem is that  $f$  is not necessarily onto. As  $L'$  is a linear order we have  $f(\beta) <_{L'} f(\delta)$  or  $f(\delta) <_{L'} f(\beta)$  (equality cannot occur by the choice of  $E$ ). Let us suppose that the former is true, the latter case is symmetric. Suppose that  $f(\beta)$  does not witness that  $i \in \text{inv}_{\bar{L}', \bar{C}}(f(\delta))$ , this then means that there is  $\gamma \in L'_{\alpha_i^\delta}$  (so  $\gamma < \alpha_i^\delta$ ) such that  $f(\beta) <_{L'} \gamma <_{L'} f(\delta)$ . Observe that there is no  $\varepsilon \in L_{\alpha_i^\delta}$  such that  $\gamma \leq_{L'} f(\varepsilon) \leq_{L'} f(\delta)$ , by the choice of  $\beta$ .

Consider  $T \stackrel{\text{def}}{=} \{x : (\neg \exists q \in L_{\alpha_i^\delta}) \gamma <_{L'} f(q) <_{L'} x\}$ . We claim that  $T \cap \alpha_i^\delta$  is exactly the set  $\{\zeta < \alpha_i^\delta : \zeta <_{L'} f(\delta)\}$ . Namely if  $\zeta < \alpha_i^\delta$  and  $\zeta <_{L'} f(\delta)$  and  $\zeta \notin T$  then there is  $\varepsilon \in L_{\alpha_i^\delta}$  such that  $\gamma <_{L'} f(\varepsilon) <_{L'} \zeta <_{L'} f(\delta)$ , a contradiction. On the other hand, if for some  $\zeta < \alpha_i^\delta$  we have  $f(\delta) <_{L'} \zeta$  then in  $M$  it is true that there is  $q$  such that  $\gamma <_{L'} f(q) <_{L'} \zeta$ , as  $\delta$  is such a  $q$ . By elementarity it is true that there is such  $q \in L_{\alpha_i^\delta}$ , so  $\zeta \notin T$ . Hence we have shown the existence of a  $\xi$  such that  $\{\zeta < \alpha_i^\delta : \zeta <_{L'} \xi\}$  is exactly

$$\{x < \alpha_i^\delta : (\neg \exists q \in L_{\alpha_i^\delta}) \gamma <_{L'} f(q) <_{L'} x\}.$$

By elementarity there must be such  $\xi \in L'_{\alpha_{i+1}^\delta} \setminus L'_{\alpha_i^\delta}$ , and then this  $\xi$  shows that  $i \in \text{inv}_{\bar{L}', \bar{C}}(f(\delta))$ . **★<sub>1.7</sub>**

To finish the proof of the theorem, suppose that there were a family  $\{L_i : i < i^*\}$  of linear orders of size  $\lambda$  for some  $i^* < 2^\kappa$ , such that every linear order of size  $\lambda$  embeds into some  $L_i$ . We may assume that the universe of each  $L_i$  is  $\lambda$ . Let  $\bar{L}_i$  be any filtration of  $L_i$  and let  $\mathcal{B}$  be the family of all  $B \subseteq \kappa$  such that for some  $x$  and  $i$  we have  $\text{inv}_{\bar{L}_i, \bar{C}}(x) = B$ . Then the size of  $\mathcal{B}$  is at most  $\lambda \cdot |i^*|$ , which is  $< 2^\kappa$ . Hence there is by Construction Lemma

a linear order  $L_A$  with universe  $\lambda$  and its filtration  $\bar{L}_A$  such that for a club  $C$  of  $\lambda$  we have  $\delta \in C \implies \text{inv}_{\bar{L}_A, \bar{C}}(\delta) = A$ . Suppose that  $f : L^* \rightarrow L_i$  is an embedding and let  $E$  be a club guaranteed to exist by the Preservation Lemma. Let  $\delta \in C$  be such that  $C_\delta \subseteq E$ . Then  $\text{inv}_{\bar{L}_i, \bar{C}}(f(\delta)) = A$ , a contradiction with the choice of  $A$ . ★<sub>1.5</sub>

## 2 A few more words on orders and orderable structures

The Kojman-Shelah method can be ramified to give results on cardinals  $\lambda$  that are not necessarily regular. Using another pcf staple, the covering number  $\text{cov}(\lambda, \mu, \theta, \sigma)$ , they were able to strap together the negative universality results on the regular cardinals below a given singular cardinal to obtain a negative universality result for the singular. Note that by classical results about special models (see [4]) there is a universal linear order (or any other first order theory of a sufficiently small size) in any strong limit uncountable cardinality. The question then becomes what happens if for example  $\aleph_\omega$  is not a strong limit. The answer is that then there is no universal linear order then. Specifically:

**Theorem 2.1** [Kojman-Shelah [15]] Suppose that  $\lambda$  is a singular cardinal which is not a strong limit and it satisfies that either

- (a)  $\aleph_\lambda > \lambda$  or
- (b)  $\aleph_\lambda = \lambda$  but  $|\{\mu < \lambda : \aleph_\mu = \mu\}| < \lambda$  and either  $\text{cf}(\lambda) = \aleph_0$  or  $2^{<\text{cf}(\mu)} < \mu$ ,

then there is no universal linear order of size  $\lambda$ .

Linear orders are representatives of theories  $T$  that have the *strict order property*, which means that there is a formula  $\varphi(\bar{x}; \bar{y})$  such that in the monster model  $\mathcal{C}$  of  $T$  there are  $\bar{a}_n$  for  $n < \omega$  such that for any  $m, n < \omega$

$$\mathcal{C} \models “(\exists \bar{x})[\neg \varphi(\bar{x}; \bar{a}_m) \wedge \varphi(\bar{x}; \bar{a}_k)]” \text{ iff } m < n.$$

Other examples of first order theories that have the strict order property are Boolean algebras, partial orders, lattices, ordered fields, ordered groups and any unstable complete theory that does not have the independence property (see [22]). Using the fact that the strict order property of  $T$  allows for coding of orders into models of  $T$  and that there is a quantifier-free definable order in the above (non-complete) theories, §5 of [15] shows that the existence of a universal element in any of these theories at a cardinal  $\lambda$  implies the existence of a universal linear order of size  $\lambda$ . Therefore the negative universality results stated above also apply to these theories.

A different approach to drawing conclusions about the universality problem in one class knowing the behaviour of another class is taken by Katherine Thompson [43] and [44], who uses functors that preserve the embedding structure. She reproves the Kojman-Shelah conclusion about universality of partial orders versus that of linear orders and connects certain classes of graphs with certain classes of strict orders. This approach is useful also when one moves from the first order context, for example to the class of orders that omit chains of a certain type. The simplest case are orders that omit infinite descending sequence. Universality is resolved trivially in the class of well orders, as follows from the example of the ordinals, but by changing the context to that of well-founded partial orders with some extra requirements one obtains a different situation. This type of problem is the subject of [43].

### 3 Model theory

There is a model-theoretic motivation behind an attempt to deliver a general method of approach to the universality problem, stemming from Shelah's programme of classification theory. This is very well described in §5 of Shelah's paper on open questions in model theory, [34]. Namely it may be hoped that the behaviour of a theory with respect to the universality would classify the theory as 'good' if it can admit a small number of universal models even when the relevant instances of  $GCH$  fail, while a bad theory would rule out small universal families as soon as  $GCH$  would be sufficiently violated (recall that the situation in the presence of  $GCH$  is information-free here, as all first order countable theories e.g. have a universal element in every uncountable

cardinal (see [4]). Such a division would be used to classify unstable theories, with the hoped for result similar to the classification of stable versus unstable theories where a model-theoretic property of stability of a countable theory was closely connected with the number of nonisomorphic models a theory may have at an uncountable cardinal, through the celebrated Shelah's Main Gap Theorem (see [22]). The idea of using universality in a similar manner has proved to be quite successful, and although no precise model-theoretic equivalent has been found as of yet, there is much information available about the existing model-theoretic properties. One can find a rather detailed description of the present state of knowledge in [9] where there is also a precise definition of the proposed division from the set-theoretic point of view, that is what is meant by being 'good' (referred to as *amenable*) and 'very-bad' (*highly non-amenable*) from the universality point of view. In this paper we mostly concentrate on the highly non-amenable theories, which can be defined by

**Definition 3.1** A theory  $T$  is said to be *highly non-amenable* iff for every large enough regular cardinal  $\lambda$  and  $\kappa < \lambda$  such that there is a truly tight  $(\kappa, \lambda)$  club guessing sequence  $\langle C_\delta : \delta \in S \rangle$  the smallest number of models of  $T$  of size  $\lambda$  needed to embed all models of  $T$  of that size is at least  $2^\kappa$ .  $T$  is *highly non-amenable up to  $\kappa^*$*  if the above characterisation is not necessarily true, but it is true whenever  $\kappa < \kappa^*$ .

In model theory one usually works with complete theories, while our examples above were not necessarily so (for example we worked with the theory of a linear order). We adopt the convention that when speaking of a complete theory by an embedding we mean an elementary embedding, and otherwise we just mean an ordinary embedding. With this clause Definition 3.1 makes sense in both contexts, and the work of Kojman and Shelah presented in §1 showed that linear orders and theories with the strict order property are highly non-amenable. Shortly after this work the same authors in [16] proceeded to show (Theorem 4.1 + Theorem 5.1 of [16]):

**Theorem 3.2** [Kojman-Shelah [16]] Countable stable unsuperstable theories are highly non-amenable up to  $\aleph_1$ . In general stable unsuperstable theories  $T$  are highly non-amenable up to their stability cardinal  $\kappa(T)$ .

Here we use the notion of the stability cardinal  $\kappa(T)$  defined as the minimal cardinal  $\kappa$  such that for every set  $A \subseteq \mathfrak{C}$  and a type  $p$  over  $A$  there is  $B \subseteq A$  such that  $|B| < \kappa$  and  $p$  does not fork over  $B$ . It is proved in Shelah's Stability Spectrum Theorem [22] that for any stable  $T$  we have  $\kappa(T) \leq |T|^+$  and for every  $\lambda$  we have  $T$  is stable in  $\lambda$  iff  $\lambda = \lambda^{<\kappa(T)}$  and either  $\lambda \geq 2^{\aleph_0}$  or  $\lambda \geq$  the number  $D(T)$  of parameter-free types of  $T$  in  $\mathfrak{C}$ . A countable complete first order theory  $T$  is stable unsuperstable iff  $\kappa(T) = \aleph_1$ . In Theorem 3.2, as well as in many other applications of the method, a major issue is how to define an invariant. Suppose that  $T$  is, for simplicity, a complete countable stable unsuperstable theory.

**Definition 3.3** Let  $\lambda$  be regular,  $\bar{C} = \langle C_\delta : \delta \in S \rangle$  a  $(\kappa, \lambda)$  tight club guessing sequence, and  $N$  a model of  $T$  of size  $\lambda$  given with a continuous increasing filtration  $\bar{N} = \langle N_i : i < \lambda \rangle$ . We define for a  $\delta \in S$  and a tuple  $\bar{a}$  of  $N$ ,

$$\text{inv}_{\bar{N}, \bar{C}}(\bar{a}) \stackrel{\text{def}}{=} \{i < \kappa : \text{the type of } \bar{a} \text{ over } N_{\alpha_i^\delta} \text{ forks over } N_{\alpha_{i+1}^\delta}\}.$$

As before, we have used the notation  $\langle \alpha_i^\delta : i < \kappa \rangle$  for the increasing enumeration of  $C_\delta$ . In the case of  $\kappa = \aleph_0$  it suffices to deal with ordinary club guessing sequences. We do not have the space to introduce the notion of forking here, but the intuition behind Definition 3.3 is similar to the idea behind the invariant for linear orders:  $i$  is in the invariant iff ‘something new happens’ at the stage  $\alpha_{i+1}^\delta$ , something that ‘reflects’ the behaviour of  $\bar{a}$  with respect to  $\bigcup_{i < \kappa} N_{\alpha_i^\delta}$ . Recalling that  $\kappa < \kappa(T)$  is assumed may give a hint of how the (rather complex) proof of [16] proceeds.

One can also try to what is meant by the ‘good’ universality behaviour, and in [9] we have tried to capture this using the notion of *amenability*.

**Definition 3.4** A theory  $T$  is *amenable* iff whenever  $\lambda$  is an uncountable cardinal larger than the size of  $T$  and satisfying  $\lambda^{<\lambda} = \lambda$  and  $2^\lambda = \lambda^+$ , while  $\theta$  satisfies  $\text{cf}(\theta) > \lambda^+$ , there is a  $\lambda^+$ -cc ( $< \lambda$ )-closed forcing notion that forces  $2^\lambda$  to be  $\theta$  and assures that in the extension there is a family  $\mathcal{F}$  of  $< \theta$  models of  $T$  of size  $\lambda^+$  such that every model of  $T$  of size  $\lambda^+$  embeds into one of the models in  $\mathcal{F}$ . Localising at a specific  $\lambda$  we obtain the definition of amenability at  $\lambda$ .

The point is that no theory can be both amenable and highly non-amenable. Namely suppose that a theory  $T$  is both amenable and highly non-amenable, and let  $\lambda$  be a large enough regular cardinal while  $V = L$  or simply  $\lambda^{<\lambda} = \lambda$  and  $\diamond(S_\lambda^{\lambda^+})$  holds. Let  $P$  be the forcing exemplifying that  $T$  is amenable. Clearly there is a truly tight  $(\lambda, \lambda^+)$  club guessing sequence  $\bar{C}$  in  $V$ , and since the forcing  $P$  is  $\lambda^+$ -cc, every club of  $\lambda^+$  in  $V^P$  contains a club of  $\lambda^+$  in  $V$ , hence  $\bar{C}$  continues to be a truly tight  $(\lambda, \lambda^+)$  club guessing sequence in  $V^P$ . Then on the one hand we have that in  $V^P$ , the universality number of models of  $T$  of size  $\lambda$ ,  $\text{univ}(T, \lambda^+)$ , is at least  $2^\lambda$  by the high non-amenableity, while  $\text{univ}(T, \lambda^+) < 2^\lambda$  by the choice of  $P$ , a contradiction. In [8], building on the earlier work of Shelah in [30], we gave an axiomatisation of elementary classes that guarantees that the underlying theory is amenable. Shelah proved in [31] that all countable simple theories are amenable at all successors of regular  $\kappa$  satisfying  $\kappa^{<\kappa} = \kappa$ . (Note that even though all simple theories are stable, this is not in contradiction with Theorem 3.2, as there it is only proved that countable stable unsuperstable theories are highly non-amenable up to  $\aleph_1$ ). In that same paper Shelah introduced a hierarchy of complexity for first order theories, and showed that high non-amenableity appears as soon as a certain level on that hierarchy is passed. Details of this hierarchy are given in the following definition:

**Definition 3.5** Let  $n \geq 3$  be a natural number. A formula  $\varphi(\bar{x}, \bar{y})$  is said to exemplify the *n-strong order property* of  $T$ ,  $SOP_n$ , if  $\text{lg}(\bar{x}) = \text{lg}(\bar{y})$ , and there are  $\bar{a}_k$  for  $k < \omega$ , each of length  $\text{lg}(\bar{x})$  such that

- (a)  $\models \varphi[\bar{a}_k, \bar{a}_m]$  for  $k < m < \omega$ ,
- (b)  $\models \neg(\exists \bar{x}_0, \dots, \bar{x}_{n-1})[\bigwedge \{\varphi(\bar{x}_\ell, \bar{x}_k) : \ell, k < n \text{ and } k = \ell + 1 \pmod n\}]$ .

The following were proved in [31]: the hierarchy above describes a sequence  $SOP_n$  ( $3 \leq n < \omega$ ) of properties of strictly increasing strength such that the theory of a dense linear order possesses all the properties, while on the other hand no simple theory can have the weakest among them,  $SOP_3$ . The property  $SOP_4$  of a theory  $T$  implies that  $T$  is highly non-amenable. In the light of these results it might then be asked if  $SOP_4$  is a characterisation

of high non-amenability, A partial solution appears in [9]. There we considered a property of theories that we called *oak property*, as its prototypical example is a tree of the form  ${}^{\kappa}\geq\lambda$  equipped with restriction where we can express that  $\eta \upharpoonright \alpha = \nu$  for  $\eta \in {}^{\kappa}\lambda$ ,  $\alpha < \kappa$  and  $\nu \in {}^{\kappa}>\lambda$ . This property is also a generalisation of the theory of infinitely many independent equivalence relations  $T_{\text{feq}}^*$ , see [9]. The formal definition is:

**Definition 3.6** A theory  $T$  is said to *satisfy the oak property as exhibited by a formula*  $\varphi(\bar{x}, \bar{y}, \bar{z})$  iff for any infinite  $\lambda, \kappa$  there are  $\bar{b}_\eta (\eta \in {}^{\kappa}>\lambda)$ ,  $\bar{c}_\nu (\nu \in {}^{\kappa}\lambda)$  and  $\bar{a}_i (i < \kappa)$  such that

- (a)  $[\eta \triangleleft \nu \ \& \ \nu \in {}^{\kappa}\lambda] \implies \varphi[\bar{a}_{lg(\eta)}, \bar{b}_\eta, \bar{c}_\nu]$ ,
- (b) If  $\eta \in {}^{\kappa}>\lambda$  and  $\eta \hat{\langle} \alpha \rangle \triangleleft \nu_1 \in {}^{\kappa}\lambda$  and  $\eta \hat{\langle} \beta \rangle \triangleleft \nu_2 \in {}^{\kappa}\lambda$ , while  $\alpha \neq \beta$  and  $i > lg(\eta)$ , then  $\neg \exists \bar{y} [\varphi(\bar{a}_i, \bar{y}, \bar{c}_{\nu_1}) \wedge \varphi(\bar{a}_i, \bar{y}, \bar{c}_{\nu_2})]$ ,

and in addition  $\varphi$  satisfies

- (c)  $\varphi(\bar{x}, \bar{y}_1, \bar{z}) \wedge \varphi(\bar{x}, \bar{y}_2, \bar{z}) \implies \bar{y}_1 = \bar{y}_2$ .

Shelah proved in [30] that  $T_{\text{feq}}^*$  exhibits a non-amenability behaviour provided that some cardinal arithmetic assumptions close to the failure of the singular cardinal hypothesis are satisfied. This does not necessarily imply high non-amenability as it was proved also in [30] that this theory is in fact amenable at any cardinal which is the successor of a cardinal  $\kappa$  satisfying  $\kappa^{<\kappa} = \kappa$ . In [9] we generalised the first of these two results by showing that any theory with oak property satisfies the same non-amenability results as those of  $T_{\text{feq}}^*$ , and we gave some more circumstances, given in terms of pcf theory for when such non-amenability results hold. The oak property cannot be made a part of the  $SOP_n$  hierarchy, as [9] gave a theory which has oak, and is  $NSOP_3$ , while the model completion of the theory of triangle free graphs is an example of a  $SOP_3$  theory which does not satisfy the oak property. On the other hand it is also proved in [9] that no oak theory is simple. Further considerations of the oak property appear in [36] where it is proved that (under an interpretation of what it means for a class to have oak) that the class of groups has this property. That paper also gives further universality results in the context of abelian groups.

## 4 Some applications in analysis and topology

There is a rich literature concerning the universality problem in the various classes of compact spaces coming from analysis, such as Corson and Eberlein compacta, where by an embedding we usually mean the existence of a continuous surjection, see e.g. [1]. Many of these questions very nicely resolved by the  $\sigma$ -functor of Todorćević, [45], which gives for every such space  $K$  another space  $\sigma(K)$  in the same class such that  $\sigma(K)$  is not a continuous image of  $K$ . The class of uniform Eberlein compacta, which are those compact spaces that are homeomorphic to a weakly compact subspace of a Hilbert space, seem to be an odd one out in these problems, since neither the method of generalised Szlenk invariants as employed in [1] nor the  $\sigma$  functor give any results in this class. The reason is, as the authors of [1] observed, that their invariant defined for all Eberlein spaces, trivialises in the case that the Eberlein is uniform, while  $\sigma(K)$  for a uniform Eberlein compact space  $K$  is an Eberlein compact but not necessarily uniform. Murray Bell in [2] made a major advance in the universality problem of the uniform Eberlein compacta, which had been completely open since the 1977 paper [3] of Yoav Benyamini, Mary Ellen Rudin and Michael Wage posed it. Namely, Bell defined a certain algebraic structure, the so called c-algebra, and he proved that there is a universal UEC of weight  $\lambda$  iff there is a universal c-algebra of size  $\lambda$ . In the same paper Bell showed that if  $2^{<\lambda} = \lambda$ , there is a c-algebra of size  $\lambda$  which is universal not just under ordinary embeddings, but also under a stronger notion of a c-embedding. We shall call such algebras *c-universal*, definitions follow. He also provided negative consistency results in models obtained by adding Cohen subsets to a regular cardinal.

**Definition 4.1** (1) A subset  $C$  of a Boolean algebra  $B$  has *the nice property* if for no finite  $F \subseteq C$  do we have  $\bigvee F = 1$ . A Boolean algebra  $B$  is a *c-algebra* iff there is a family  $\langle A_n : n < \omega \rangle$  of pairwise disjoint subsets of  $B$  each consisting of pairwise disjoint elements, whose union has the nice property and generates  $B$ .

(2) If  $B_l$  for  $l \in \{0, 1\}$  are c-algebras with fixed sequences  $\langle A_n^l : n < \omega \rangle$  of subsets exemplifying that  $B_l$  is a c-algebra, then a 1-1 Boolean homomorphism  $f : B_0 \rightarrow B^*$  is a *c-embedding* iff  $f''A_n^l \subseteq A_n^l$  for all  $n < \omega$ .



Note that the notion of a c-algebra is not first order so the Kojman-Shelah results from [15] do not directly apply. We showed in [6] that for no regular cardinal  $\lambda > \aleph_1$  with  $2^{\aleph_0} > \lambda$  can there exist  $< 2^{\aleph_0}$  c-algebras of size  $\lambda$  such that every c-algebra of size  $\lambda$  embeds into one of them. These results were continued in [7] which contains both negative and positive results about the existence of universal c-algebras and UEC. On the other hand, we proved a positive consistency result showing that under certain non-GCH assumptions there can be a family of UEC of a relatively small size ( $\lambda^{++} < 2^{\lambda^+}$ ) each of which has weight  $\lambda^+$  and which are jointly universal for UEC of weight  $\lambda^+$ . The negative results are the ones relevant to this paper, and they were obtained using the method of Kojman-Shelah invariants. The appropriate definition in this context turned out to be the following:

**Definition 4.2** Let  $\lambda$  be a regular cardinal and  $\langle C_\delta : \delta \in S \rangle$  a club guessing sequence on  $\lambda$ , with  $C_\delta = \langle \alpha_i^\delta : i < i^* \rangle$  an increasing enumeration. Let  $B$  be a c-algebra of size  $\lambda$  with a filtration  $\bar{B}$ , and we assume that  $\langle A_n : n < \omega \rangle$  is a fixed sequence demonstrating that  $B$  is a c-algebra. Suppose that  $\delta \in S$  and define for  $\delta \in S$  and  $b \in B \setminus B_\delta$

$$\text{inv}_{\bar{B}, \bar{C}}(b) \stackrel{\text{def}}{=} \left\{ i < i^* : (\exists m \geq 1)(\exists y \in A_m \cap B_{\alpha_{i+1}^\delta} \setminus B_{\alpha_i^\delta}) [y \geq b] \right\}.$$

It is interesting to note that the mutual exclusiveness of amenability and high non-amenability as defined in §3 does not apply if these definitions are taken in their obvious translation to the non-first order context, and that the class of c-algebras exemplifies that.

A recent application of the method of invariants comes from Kojman-Shelah's work on almost isometric embeddings between metric spaces, [18]. A map  $f : X \rightarrow Y$  between metric spaces is said to be *Lipshitz with constant*  $r > 0$  if for every  $x, y \in X$  we have  $d_Y(f(x), f(y)) < r \cdot d_X(x, y)$ .  $X$  is *almost isometrically embeddable into*  $Y$  iff for every  $r > 1$  there is a continuous injection  $f : X \rightarrow Y$  such that both  $f$  and  $f^{-1}$  are Lipshitz with constant  $r$ , which is called *bi-Lipshitz with constant*  $r$ . Among many interesting results about such embeddings, [18] also gives

**Theorem 4.3** [Kojman-Shelah, [18]] If  $\aleph_1 < \lambda < 2^{\aleph_0}$  is regular then for every  $\kappa < 2^{\aleph_0}$  and metric spaces  $\{(X, d_i) : i < \kappa\}$  of size  $\lambda$ , there exists a

metric space of size  $\lambda$  that is not almost isometrically embeddable into any  $(X, d_i)$ .

The proof of the theorem again uses the method of invariants, but with a twist. Namely one defines two kind of invariants,  $\text{inv}^{\text{dom}}$  and  $\text{inv}^{\text{rng}}$ , as follows, where we are using the same notation for club guessing sequences as above:

**Definition 4.4** Suppose that  $(X, d)$  is a metric space with universe  $\lambda$ ,  $\bar{C}$  is an  $S_{\aleph_0}^\lambda$  club guessing sequence,  $\delta \in S_{\aleph_0}^\lambda$ ,  $\beta > \delta$  and  $K \geq 1$  is an integer. We consider  $X$  as being given in the filtration  $\bar{X} = \{\alpha : \alpha < \lambda\}$ . Then

$$\begin{aligned} \text{inv}_{\bar{C}, X, \delta}^{\text{dom}}(\beta) &= \{n < \omega : d(\beta, \alpha_n^\delta)/d(\beta, \alpha_{n+1}^\delta) > 2K^2\} \text{ and} \\ \text{inv}_{\bar{C}, X, \delta}^{\text{ran}}(\beta) &= \{n < \omega : d(\beta, \alpha_n^\delta)/d(\beta, \alpha_{n+1}^\delta) > 4K^4\}. \end{aligned}$$

The Preservation Lemma then says in particular that if  $f : X \rightarrow Y$ , where both  $X$  and  $Y$  are metric spaces with universe  $\lambda$ , is bi-Lipshitz with constant  $K$ , then there is a club  $E$  of  $\lambda$  such that for every  $\delta \in E \cap S_{\aleph_0}^\lambda$  and  $\beta > \delta$ , we have  $f(\beta) > \delta$  and  $\text{inv}_{\bar{C}, X, \delta}^{\text{dom}}(\beta) = \text{inv}_{\bar{C}, Y, \delta}^{\text{rng}}(f(\beta))$ . Theorem 4.3 is to be contrasted with another theorem from [18], which says that for any regular cardinal  $\lambda$  it is consistent that  $2^{\aleph_0} > \lambda^+$  and there  $\lambda^+$  separable metric spaces of size  $\lambda$  such that every separable metric space of size  $\lambda$  almost-isometrically embeds into one of them. Earlier results about universality of metric spaces under different kinds of embeddings and involving the method of invariants were obtained by Shelah in [32].

Model theory of metric spaces is also one of the subjects of Alex Ustvyasov's Ph.D thesis [42] and his joint work with Shelah in [40], where they concentrate on complete metric spaces. A model-theoretic approach to Banach spaces pays off in Shelah-Ustvyasov's paper [39] where they prove that the appropriately axiomatised theory of Banach spaces has  $SOP_n$  for all  $n \geq 3$ , and hence draw the negative universality results provided by  $SOP_4$  (see §3), where the notion of embedding is isometry. Note that if  $\lambda = 2^{<\lambda} > \aleph_0$  then there is an isometrically universal Banach space of size  $\lambda$ . Universality results in Banach spaces are quite well studied classically, maybe the most well known result in this vein is that of Szlenk in [41] who proved that there is no universal reflexive separable Banach space.

## 5 Some applications in algebra

A very fruitful application of the Kojman-Shelah method of invariants has been in the theory of infinite abelian groups, which we shall take in their additive notation. In [17] Kojman and Shelah study the problem of universality in several kinds of groups under various kinds of embeddings. Many classes of groups simply have a universal element under ordinary embeddings in every infinite cardinality, namely there is always a universal group, universal  $p$ -group (for any prime  $p$ ), universal torsion group and universal torsion-free group (see [17]). On the other hand there is no universal reduced  $p$ -group. The situation becomes different when one restricts the kind of embeddings and the kind of groups one considers. Of particular interest are *pure* embeddings, where a group monomorphism  $f : H \rightarrow G$  is a pure embedding if  $f''H$  satisfies that for all  $n < \omega$ ,  $nf''H = nG \cap f''H$ . In other words,  $f''H$  is a *pure* subgroup of  $G$ .

The appropriate notion of the invariant here is

**Definition 5.1** Suppose that  $\lambda > \aleph_0$  is regular cardinal,  $G$  is an abelian group of size  $\lambda$  given with its filtration  $\bar{G}$  and  $\langle C_\delta : \delta \in S \subseteq \lambda \rangle$  is a club guessing sequence on  $\lambda$  where for each  $\delta$  the increasing enumeration of  $C_\delta$  is  $\langle \alpha_i^\delta : i < i_\delta^* \rangle$ . For  $g \in G$  and  $\delta \in S$  we define

$$\text{inv}_{\bar{G}, \bar{C}, \delta}(g) \stackrel{\text{def}}{=} \{i < i_\delta^* : g \in \bigcup_{n < \omega} ((G_{\alpha_{i+1}^\delta} + nG) \setminus (G_{\alpha_i^\delta} + nG))\}.$$

A Preservation Lemma can be proved for this type of invariants and pure embeddings. The paper gives a number of constructions of various types of groups with the prescribed invariant, which allows for the proof of several theorems, a selection of which is:

**Theorem 5.2** [Kojman-Shelah, [17]] Suppose that  $\lambda$  is regular and for some  $\mu, \mu^+ < \lambda < 2^\mu$ , while  $p$  is any prime. Then there is no

- (a) purely universal separable  $p$ -group of size  $\lambda$ ;
- (b) universal reduced slender group of size  $\lambda$ ;
- (c) universal reduced torsion-free group of size  $\lambda$ .

Research on the universality in various classes of groups was continued by Shelah in [29], [32] and [33], where he considered various classes of groups under ordinary embeddings (so they are not assumed to be pure). In [29] the class considered is that of  $(< \lambda)$ -stable abelian groups, which means that for every subset  $A$  of  $G$  of size  $< \lambda$  the closure in  $G$  of the subgroup  $\langle A \rangle_G$  generated by  $A$ , defined as

$$\text{cl}_G(\langle A \rangle_G) = \{x : \inf_{y \in \langle A \rangle_G} (\min_{i > 1} \{2^{-i} : x - y \text{ divisible by } \prod_{1 < j < i} n_j\}) = 0\}$$

for some conveniently chosen and fixed increasing sequence  $\langle n_i : i < \omega \rangle$  of natural numbers  $> 1$ . This notion in particular includes strongly  $\lambda$ -free groups and it can be handled using the same definition of invariant as in Definition 5.1. In [32] there is a deep analysis of how necessary this is, and it proceeds through a series of results about classes of trees with  $\omega + 1$  levels, with the thesis that these are a prototype for various classes of groups (deriving also some surprising results as to what kind of trees one needs to look at here). Of particular interest in [32] are reduced torsion free groups and reduced separable abelian  $p$ -groups, but the paper indeed gives a very rich selection of results on various classes of both groups and trees. This research was continued in [33] and [38], and as a combined result of, one has almost a complete calculation of the universality spectrum of the reduced torsion free abelian groups and reduced separable  $p$ -groups, for example:

**Theorem 5.3** [Shelah] Let  $\lambda$  be an infinite cardinal and  $\mathcal{K}_\lambda$  the class of reduced torsion free abelian groups of size  $\lambda$  considered under ordinary embeddings.

- (a) If  $\lambda = \lambda^{\aleph_0}$  or  $\lambda$  is singular of countable cofinality and  $(\forall \theta < \lambda) \theta^{\aleph_0} < \theta$ , then there is a universal member of  $\mathcal{K}_\lambda$ .
- (b) If  $\lambda < 2^{\aleph_0}$ , or for some  $\mu$  we have  $\beth_\omega + \mu^+ < \lambda = \text{cf}(\lambda) < \mu^{\aleph_0}$ , then there is no universal member of  $\mathcal{K}_\lambda$ .

Some of the remaining cases of possible cardinal arithmetic assumptions were reduced to some weak pcf assumptions the consistency of whose failure is not known. In [32] there are also results about modules. Model theoretic

properties of groups related to universality are studied in [39], where it is proved that if  $\mathcal{G}$  is the “universal domain” (a monster model for groups) then it has  $SOP_3$  and, surprisingly, it does not have  $SOP_4$ .

## 6 Some applications in graph theory and a representation theorem

Universality problem in the class of graphs has a particularly long tradition, see for example the well known Richard Rado’s paper [21] with the construction of the Rado graph. If one considers graphs with the ordinary notion of embedding (so the edges are kept, but not necessarily non-edges, also called weak embedding), then under  $GCH$  there is a universal object in every infinite cardinality, as follows for uncountable cardinalities from the classical first order model theory and was also proved independently by Rado. Similar results hold for the class of graphs omitting the complete graph  $K_n$  of size  $n$ , where  $n \geq 3$ . It is a very interesting result of Shelah that even when  $CH$  fails there can be a universal graph of size  $\aleph_1$ , see [26], [27]. Results in [31] and [8] imply that the theory of graphs is amenable. The situation becomes different when one restricts to graphs that omit a certain structure. For example, the model completion of the theory of triangle-free graphs is amenable but the model completion of the theory of directed graphs omitting directed cycles of length  $\leq 4$  has  $SOP_4$  and is hence highly non-amenable (see [31], where a number of other similar results is given). Passing to graphs that omit an infinite structure, so exiting the realm of the first order theories, the situation immediately becomes very different. For example, it is a mathematical folklore (see [19]) that there is no universal  $K_{\aleph_0}$ -free graph in any cardinality. Komjáth and Shelah in [19] investigate the class of  $K_\kappa$ -free graphs and show that under  $GCH$  a universal exists in  $\lambda$  iff  $\kappa$  is finite or  $\text{cf}(\kappa) > \text{cf}(\lambda)$ . They also give consistency results showing how much the universality number of this class can be when it is known that there is no one universal element. There is a very rich literature available on the problem of the existence of a universal member in various classes of graphs, for example there is a complete classification of countable homogeneous directed graphs and countable

homogeneous  $n$ -tournaments, obtained by Gregory Cherlin in the memoir [5]. We cannot even begin to do justice to this rich literature in this survey, so we shall simply concentrate on the impact the club guessing method has had. This will also give us an opportune way of closing this paper by a theorem which very elegantly shifts the method of invariants from an arbitrary class of models of size  $< 2^{\aleph_0}$  to a consideration of the structure of the subsets of the reals, namely a Representation Theorem by Kojman.

A *ray* in a graph is a 1-way infinite path. A *tail* of such a ray is any infinite connected subgraph, and two rays are *tail equivalent* if they have a common ray. Consider the class  $\mathcal{K}$  of graphs  $G$  that satisfy that for every vertex  $v$  of  $G$  the induced subgraph of  $G$  spanned by  $v$  has at most one ray, up to tail equivalence. This class can also be described in terms of forbidding certain structures. Among other theorems that Kojman proves about this class in [14] is that for a regular uncountable  $\lambda$  the smallest size of a family of graphs in  $\mathcal{K}$  of size  $\lambda$  (denoted by  $\mathcal{K}_\lambda$ ) needed to embed such graphs is at least  $2^{\aleph_0}$ . This theorem actually follows from a representation obtained in the following

**Theorem 6.1** [Kojman’s Representation Theorem, [14]] If  $\lambda > \aleph_1$  is regular then there is a surjective homomorphism from the structure  $\mathcal{K}_\lambda$  partially ordered by the embedding relation, to the structure  $[\mathbb{R}]^{\leq \lambda}$  partially ordered by the subset relation.

The method of invariants is still used here, where the Construction Lemma corresponds to proving that the proposed map is surjective and the Preservation Lemma corresponds to showing that the map is a homomorphism. The Representation Theorem has some advantages over the Construction and Preservation approach because it allows for a smooth way to handle singular cardinals. In an upcoming paper [10] we have used this method to consider well-founded partial orders under rank-preserving embeddings and some other classes, and to prove negative universality results analogous to those in [14].

Let us finish by mentioning that guessing sequences stronger than club guessing are used in a recent paper of Shelah [35] to obtain negative results about the universality of the class of graphs that omit complete bipartite

graphs. The paper also gives a complete characterisation of the universality problem in this class under  $GCH$ .

## 7 Some questions

There are many open questions in this subject, and considering instances of universality in a specific class is an interesting pursuit per se. We have selected two more general questions that are in our view very important. The first is in model theory:

**Question 7.1** Does  $SOP_4$  characterise high non-amenability, in other words does every highly non-amenable theory has the  $SOP_4$  property?

The interest of this question is described in §3. The second question is in set theory and calls for a finer understanding of our forcing iteration techniques. Namely, the reader may have noticed that the definition 3.4 does not refer to the existence of universal family  $\mathcal{F}$  of size 1, namely the universal model. The reason is that all we know how to do, in the generality of the axioms of [8] or in specific forcing proofs of universality such as [20] (where Alan Mekler and Jouko Väänänen produced consistently with  $CH$  a family of  $\aleph_2$  trees of size  $\aleph_1$  with no uncountable branches and universal under reductions) or the Kojman-Shelah's theorem about separable metric spaces mentioned above, is to produce  $\lambda^+$  models of size  $\lambda$  jointly universal for models of size  $\lambda$ . Hence the question is:

**Question 7.2** Suppose that a theory  $T$  is amenable,  $\lambda$  is an uncountable cardinal larger than the size of  $T$  and satisfying  $\lambda^{<\lambda} = \lambda$  and  $2^\lambda = \lambda^+$ , while  $\theta$  satisfies  $\text{cf}(\theta) > \lambda^+$ . Can one find a cardinality preserving forcing extension in which  $2^\lambda = \theta$  and  $T$  has a universal model of size  $\lambda^+$ ?

## References

- [1] S. Argyros and Y. Benyamini, *Universal WCG Banach Spaces and Universal Eberlein Compacts*, Israel Journal of Mathematics, vol. 58 (1987), 305-320.
- [2] M. Bell, *Universal uniform Eberlein compact spaces*, Proceedings of AMS, vol. 128 (7) (2000), 2191-2197.
- [3] Y. Benyamini, M. E. Rudin, and M. Wage, *Continuous images of weakly compact subsets of Banach spaces*, Pacific Journal of Mathematics, vol. 70 (1977), 309-324.
- [4] C. Chang and J. Keisler, *Model Theory*, 2nd. ed, North-Holland, 1990.
- [5] G. Cherlin, *The classification of countable homogeneous directed graphs and countable homogeneous  $n$ -tournaments*, AMS Memoir vol. 131, no. 621 (1998).
- [6] M. Džamonja, *On uniform Eberlein compacta and  $c$ -algebras*, Topology Proceedings, vol. 23, (1998), 143-150.
- [7] M. Džamonja, *Universality of Uniform Eberlein Compacta*, submitted to the Proceedings of AMS.
- [8] M. Džamonja and S. Shelah, *On the existence of universals*, Archive for Mathematical Logic, vol. 43 (2004), 901-936.
- [9] M. Džamonja and S. Shelah, *On properties of theories which preclude the existence of universal models*, submitted to the Annals of Pure and Applied Logic.
- [10] M. Džamonja and K. Thompson, *Universality results for well-founded posets*, preprint August 2004.
- [11] M. Foreman and P. Komjáth, *The Club Guessing Ideal (Commentary on a theorem of Gitik and Shelah)*, preprint February 2003.



- [12] Z. Füredi and P. Komjáth, *Nonexistence of universal graphs without some trees*, *Combinatorica*, vol. 17 (1997), 163-171.
- [13] T. Ishiu, *Club Guessing Sequences and Filters*, Ph. D. thesis, University of California at Irvine 2003, available at <http://www.math.ukans.edu/~ishiu/research.html>
- [14] M. Kojman, *Representing embeddability as set inclusion*, *Journal of the London Mathematical Society* (2nd series), vol 58, no. 185, Part 2 (1998), 257-270.
- [15] M. Kojman and S. Shelah, *Non-existence of Universal Orders in Many Cardinals*, *Journal of Symbolic Logic*, vol. 57 (1992), 875-891.
- [16] M. Kojman and S. Shelah, *The universality spectrum of stable unsuperstable theories*, *Annals of Pure and Applied Logic*, vol. 58 (1992), 57-72.
- [17] M. Kojman and S. Shelah, *Universal Abelian Groups*, *Israel Journal of Mathematics*, vol. 92 (1995), 113-124.
- [18] M. Kojman and S. Shelah, *Almost isometric embeddings between metric spaces*, preprint, November 2003.
- [19] P. Komjáth and S. Shelah, *Universal graphs without large cliques*, *Journal of Combinatorial Theory. Ser. B*, vol. 63 (1995), 125-135.
- [20] A. Mekler and J. Väänänen, *Trees and  $\Pi_1^1$  subsets of  ${}^{\omega_1}\omega_1$* , *Journal of Symbolic Logic* 58 (3) (1993), 1052-1070.
- [21] R. Rado, *Universal graphs and universal functions*, *Acta Arithmetica*, vol. 9 (1964), 331-340.
- [22] S. Shelah, *Classification theory and the number of nonisomorphic models*, vol. 92 of *Studies in Logic and the Foundation of Mathematics*, North-Holland, Amsterdam, 1990.

- [23] S. Shelah, *Proper and Improper Forcing*, 2nd. ed., Perspectives in Mathematical Logic, Springer-Verlag 1998.
- [24] S. Shelah, *Cardinal Arithmetic*, Oxford University Press 1994.
- [25] S. Shelah, *Independence Results*, Journal of Symbolic Logic, vol. 45 (1980), 563-573.
- [26] S. Shelah, *On universal graphs without instances of CH*, Annals of Pure and Applied Logic, vol. 26 (1984), 75-87.
- [27] S. Shelah, *Universal graphs without instances of CH: revisited*, vol. 70 (1990), 69-81.
- [28] S. Shelah, *Advances in Cardinal Arithmetic*, in *Proceedings of the Banff Conference in Alberta; 4/91*, NATO Conference on Finite and Infinite Combinatorics, 1993.
- [29] S. Shelah, *Universal in  $< \lambda$ -stable abelian group*, Mathematica Japonica, vol. 43 (1996), 1-11.
- [30] S. Shelah, *The Universality Spectrum: Consistency for more classes in Combinatorics, Paul Erdős is Eighty*, Bolyai Society Mathematical Studies, vol. 1, (1993), 403-420 (proceedings of the Meeting in honor of P. Erdős, Keszthely, Hungary 7. 1993, an improved version available at <http://www.math.rutgers.edu/~shelarch>).
- [31] S. Shelah, *Toward classifying unstable theories*, Annals of Pure and Applied Logic, vol. 80, (1996), 229-255.
- [32] S. Shelah, *Non-existence of universals for classes like reduced torsion free abelian groups under embeddings which are not necessarily pure*, in *Advances in Algebra and Model Theory* M. Droste and R. Goebel (eds), Algebra, Logic and Applications, vol. 9 (1997), 229-286.
- [33] S. Shelah, *Non-existence of universal members in classes of Abelian groups*, Journal of Group Theory, vol. 4 (2001), 169-191.

- [34] S. Shelah, *On what I do not understand (and have something to say), model theory*, *Mathematica Japonica*, vol. 51 (2000), 329-377.
- [35] S. Shelah, *Is there universality among the graph omitting a complete bipartite graph*, preprint March 2004.
- [36] S. Shelah, *Existence of universal structures*, in preparation.
- [37] S. Shelah, *Analytical Guide and Corrections to [24]*, <http://www.math.rutgers.edu/~shelarch>
- [38] S. Shelah, *Notes on abelian groups*, in preparation.
- [39] S. Shelah and A. Usvyatsov, *Banach spaces and groups- order properties and universal models*, preprint February 2002.
- [40] S. Shelah and A. Usvyatsov, *Model theoretic stability for complete metric spaces*, in preparation.
- [41] W. Szlenk, *The non-existence of a separable reflexive Banach space universal for all separable reflexive Banach spaces*, *Studia Mathematica*, vol. 30 (1968), 53-61.
- [42] A. Usvyatsov, Ph.D. thesis Hebrew University of Jerusalem, in preparation.
- [43] K. Thompson, *Universality results for ordered and directed structures*, Ph.D. thesis, University of East Anglia 2003.
- [44] K. Thompson, *A universality mishmash*, preprint August 2004.
- [45] S. Todorćević, *The functor  $\sigma^2 X$* , *Studia Mathematica*, vol. 116, no. 1, (1995) 49–57.